ms45792 // John Marshall (St Andrews alumnus, lecturer in mathematics) to D'Arcy Wentworth Thompson// 23-Nov-1914

*text in red – could not properly read

(By D'Arcy Wentworth Thompson):

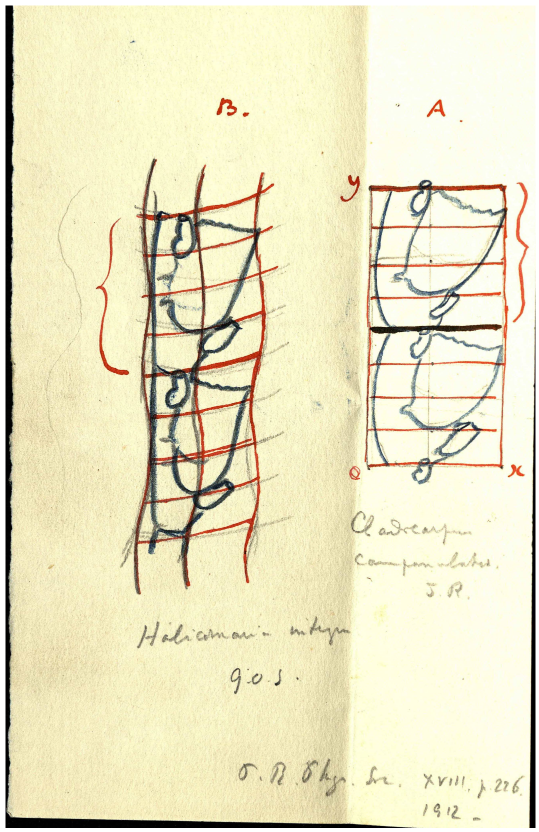

Figure 1: by D'Arcy Wentworth Thompson

We have an organism consisting of a series of segments indefinitely repeated. Fig A shows two such segments inscribed in regular coordinates. In (other or allied?) species (eg Fig B) the (?) ordinates a reflection by sinuous lines, which are evidently to be duplicated by the amplitude and phase of a sin-curve. While the (?) abscissa and replace by curved lines (of undefined but identical curvature) set at a (?) (?) (?) to the new ordinates.

In transforming A into B, thus we want an expansion, (compared?) to $A = b(xy)$ which shall include:

1. The sinuous form for the ordinates
2. // curved // // // abscissa
3. // altered distances apart of the ordinates (?)
4. // conversion of the rectangular system with an oblique one

Question (in red): Will it, or not, be the case, that, if we had begun by drawing B instead of A in regular coordinates then the expansion for its transformation with A would be the same, only with change of sign, that we have got for the curving of A into B.

Answer (by Marshall): It will not in general, be true that the transformation which makes $A \to B$ will make starting with B, $B \to A$. As I have tried to indicate in some of cases.

(pg1) Cases (1) and (2) are solvable in terms of conjugate functions provided in the transformation the new network has its lines cutting at the right angles and that there is a one to on correspondence of the pts in A and B, i.e Uniformal Representation.

The general method is this if

$$u + iv = \phi(x + iy) \tag{1}$$

where i is the imaginary quantity $\sqrt{-1}$.

If we can split up ϕ into real and imaginary parts f and ψ are functions of x and y and

$$\phi(x + iy) = f(x, y) + i\psi(x, y) = u + iv \tag{2}$$

then

$$u = f(x, y) \tag{3}$$

$$v = \psi(x, y) \tag{4}$$

gives transformation such that lines $u = const$, $v = const$, cut at right angles. Special transformations: Let

$$u + iv = (x + yi)^2 = x^2 + (iy)^2 + 2ixy = x^2 - y^2 + 2ixy \tag{5}$$

$$u = x^2 - y^2 \tag{6}$$

$$v = 2xy \tag{7}$$

(pg2)

$$x + iy = (u + iv)^2 \tag{8}$$

$$x = u^2 - v^2 \tag{9}$$

$$y = 2uv \tag{10}$$

Eliminate u: $y^2 = 4v^2(x + v^2)$

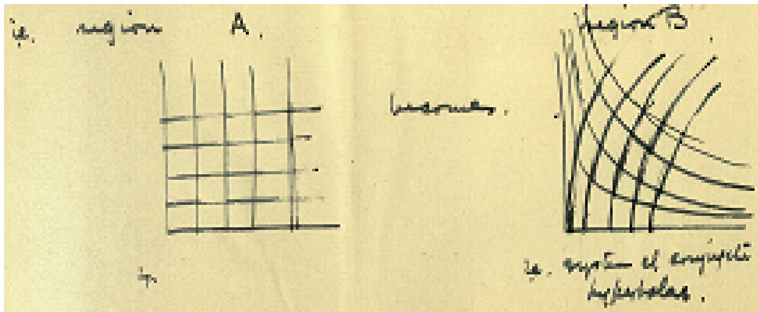

Figure 2: by Marshall

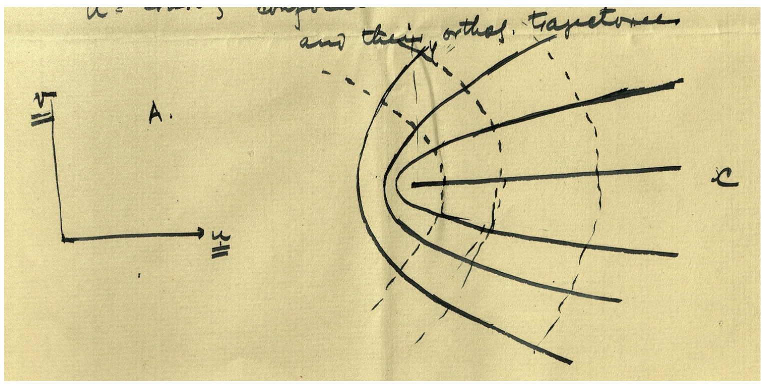

Figure 3: by Marshall

Therefore,$v = const$ and $u = const$ – are system of (?) (?) parabolas and their orthogonal trajectories.

(pg3)

$$x + iy = \frac{1}{(u+iv)} = \frac{(u-iv)}{(u^2+v^2)} \tag{11}$$

$$x = \frac{u}{(u^2+v^2)} \tag{12}$$

$$y = \frac{-v}{(u^2+v^2)} \tag{13}$$

x constant gives $u^2 + v^2 = (const)u$
y constant gives $u^2 + v^2 = -(const)v$
i.e. system of circles cutting each other as indicated.

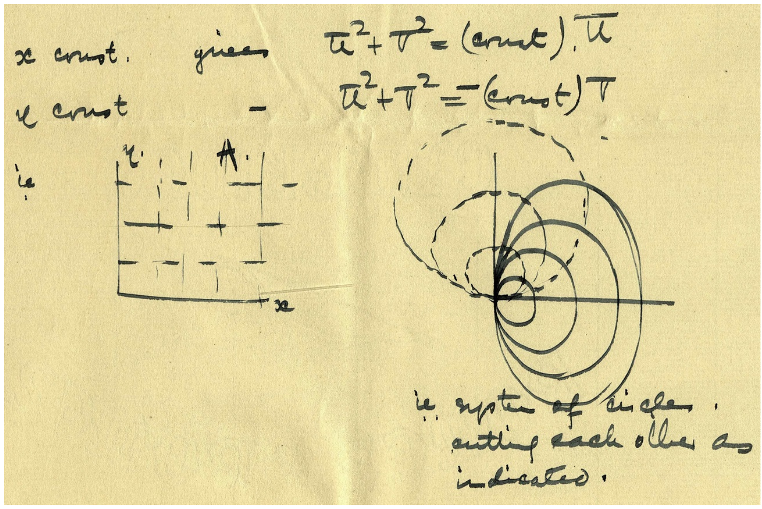

Figure 4: by Marshall

$$u + iv = \log(x + iy) = \log(re^{i\theta}) = \log(r) + i\theta \tag{14}$$

(*not sure if it's r, x or y in the equation above)

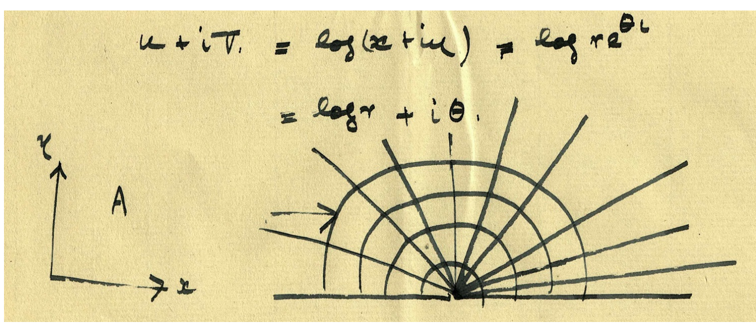

Figure 5: by Marshall

(pg4)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{15}$$

$$x + iy = a\cos(u + iv) + ib\sin(u + iv) \tag{16}$$

Put $a = e\cosh(\alpha)$ and $b = e\sinh(\alpha) \rightarrow \cosh(\alpha) = \frac{e^{\alpha}+e^{-\alpha}}{2}$

$$x + iy = e\cos(u + i(v + \alpha)) = e\cosh(u' + iv') \tag{17}$$

by change of constants

$$x = e\cosh(u')\cos(v') \tag{18}$$

$$y = e\sinh(u')\sin(v') \tag{19}$$

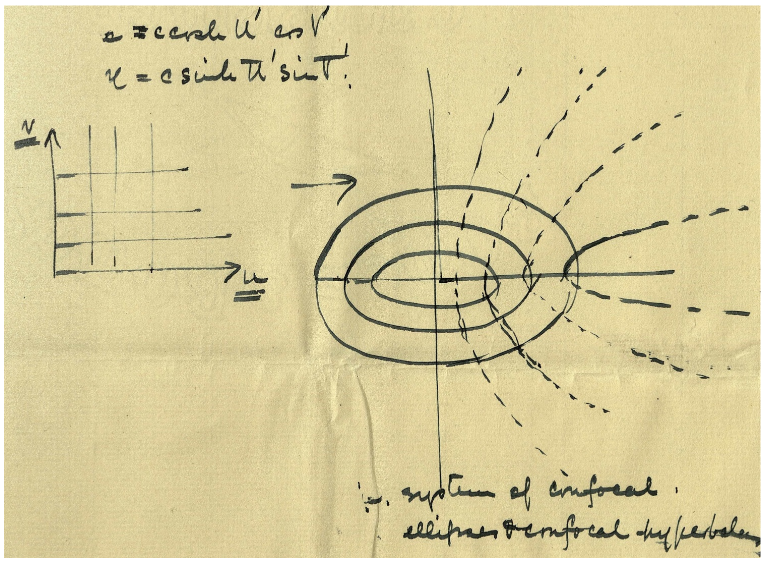

Figure 6: by Marshall

i.e. system of (?) ellipses and (?) hyperbolas

(pg5) (3) is the easiest case:

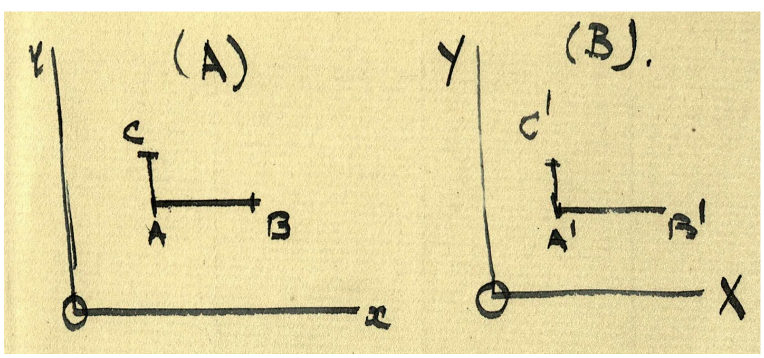

Figure 7: by Marshall

Let

$$\frac{AB}{A'B'} = \lambda \tag{20}$$

$$\frac{AC}{A'C'} = \mu \tag{21}$$

$x = \lambda X \rightarrow$ gives the transformation required
$y = \mu Y \rightarrow //$
i.e. $f(x, y)$ =constant represents curve in (A) – (1)
$f(\lambda X, \mu Y)$ = constant represents curve in (B) –(2)
where each abscissa has been aliened in ratio λ:1
and each ordinate as been aliened in ratio μ:1

If we now put $X = \lambda x$ and $Y = \mu y$
Then $f(\lambda X, \mu Y) = const$ in B
becomes $f(\lambda^2 x, \mu^2 y) = const$ in A, which is different from initial form (1)
In order to convert B back to A we require: $X = \frac{x}{\lambda}$ and $Y = \frac{y}{\mu}$ as is obvious.

(pg 6) (4) If you mean P goes to P' as in fig (in the Figure 8)

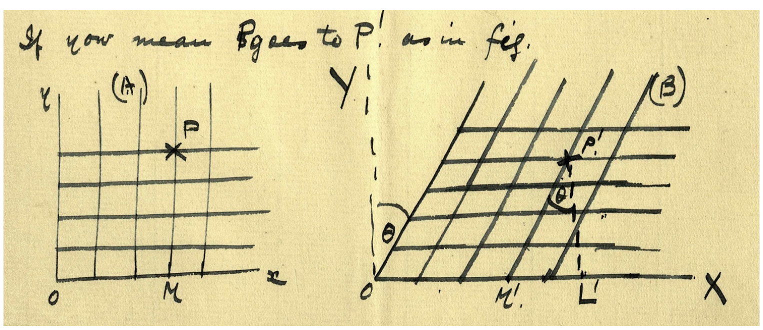

Figure 8: by Marshall

i.e. $\frac{OM}{MP} = \frac{OM'}{M'P'}$ say $OM = OM' = x$ and $MP = M'P' = y$ $X = OL' = OM' + M'L' = x + y\sin(\theta)$ — (1) is transf B to A
$Y = L'P' = y\cos(\theta)$ — (1) is transf B to A
i.e. $x = X - Y\tan(\theta)$ – is transformation A to B
$y = Y\sec(\theta)$ – is transformation A to B
i.e. $f(x, y)$ becomes $f(X - Y\tan(\theta), Y\sec(\theta))$ It is obvious that application again of this transformation will not give the (A) figure from the (B). The proper transformation is given in (1).
