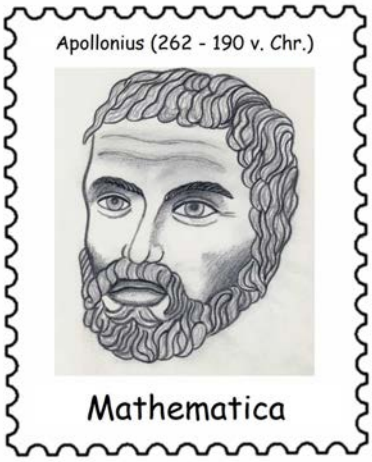
APOLLONIUS OF PERGA (262 – 190 BC)

by HEINZ KLAUS STRICK, Germany

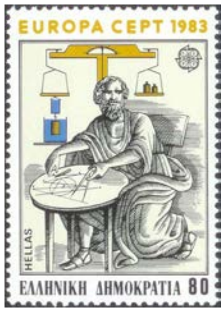


(drawing © Andreas Strick)

The fact that the Greek postal administration has so far only remembered the mathematicians THALES (624 – 547 BC), PYTHAGORAS (580 – 500 BC) and ARCHIMEDES (287 – 212 BC), but forgotten all others, such as EUCLID of Alexandria (360 – 280 BC) or APOLLONIUS OF PERGA, cannot be solely due to the fact that there are no authentic portraits of these personalities.

When, around the year 800, Islamic scholars in Baghdad began to collect the original Greek writings that still existed and translate them into Arabic, it was above all the works of EUCLID, ARCHIMEDES and APOLLONIUS that they sought to preserve.

All that was known about the famous mathematician APOLLONIUS (the addition of Perga helps to distinguish him from other Greek scholars of the same name), was that he was born in Perga (near the present-day city of Antalya) and that he went to study in Alexandria, where he later worked as a teacher, and where he died. In the meantime he taught at the University of Pergamon (near the present city of Izmir).



Our knowledge about the mathematician, of whom DESCARTES and NEWTON speak as "the great geometrician", we get from the prefaces of the different books of his probably most famous work *Conics*, of which only the first four of the total eight books (chapters) are preserved in the original.

Around the year 800, Books V to VII of *Conics* also existed and they were translated by Islamic scholars. Volume VIII was considered lost even then.

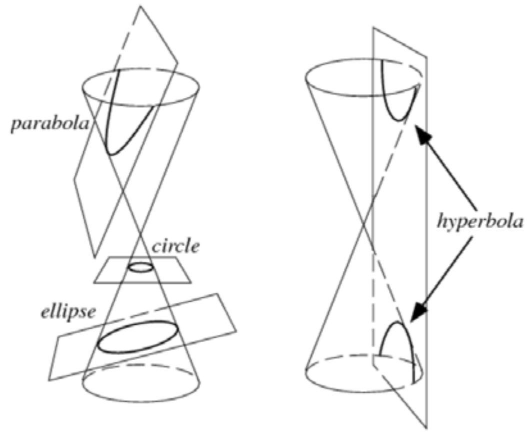
IBN AL-HAITHAM (965 – 1039) was the first to attempt a reconstruction based on the references in the writings of PAPPUS, the last important Greek mathematician (around 300 AD).

In 1710 EDMOND HALLEY published a complete edition of *Apollonii Conicorum libri octo* (books I to IV in the original, books V to VII in Latin translation, book VIII in the version he reconstructed).

ARCHIMEDES and EUCLID had already dealt with the curves that arise when a plane cuts through a cone, but the cones they considered had been created by rotating a right-angled, acute-angled or obtuse-angled triangle, and the cut was made perpendicular to the cone-producing side of the triangle.

APOLLONIUS, on the other hand, examines sections on any double cone. These double cones are generated by straight lines passing through a base circle (lying in a plane) and a fixed point (the *apex* of the cone) outside the plane. Depending on the angle of intersection, one and the same double cone will produce different intersection curves.

If the intersection is parallel to a generating line, a parabola is produced; if the plane intersects both parts of the double cone, two hyperbolas are produced (only in the 17th century are they called the two branches of a hyperbola), otherwise we get an ellipse (in special cases a circle).



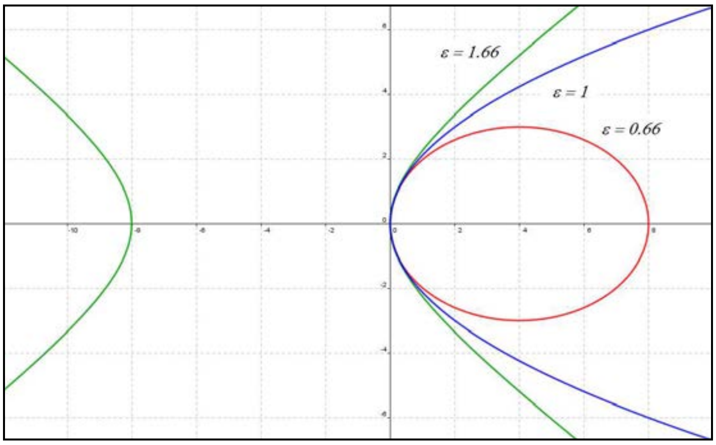
(source: mathworld.wolfram.com)

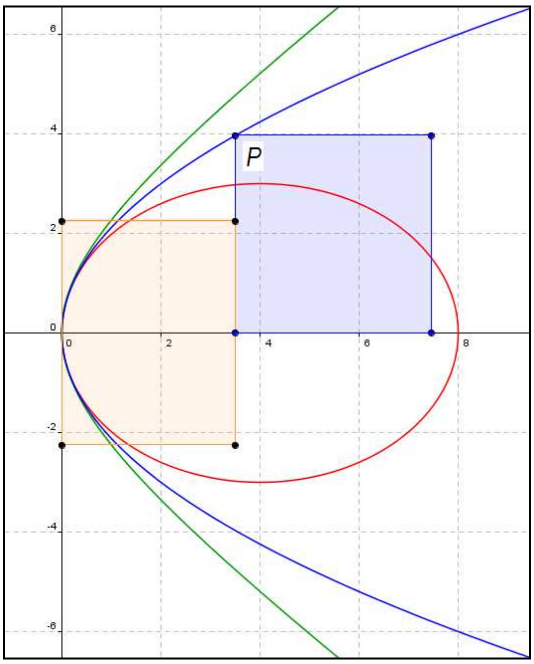
Thus APOLLONIUS created a unified theory of conic sections, which later became the model for DESCARTES' *Analytical Geometry*.

In Book I of the *Conics*, APOLLONIUS clarified the basic concepts and the characteristic properties (the so-called *symptoma*), which is what makes the curves different. He proved that the centres of mutually parallel chords lie on a straight line, the *diameter* of the conic section. Sentence 1 of Book I stated that the *symptoma* applied regardless of the choice of diameter.

Today it is common practice to describe conic sections by a common CARTESian equation:

$y^2 = 2px - (1-\varepsilon^2)\cdot x^2$ where ε is referred to as the *eccentricity* and for $\varepsilon > 1$ we get a hyperbola, for $\varepsilon = 1$ a parabola and for $\varepsilon < 1$ an ellipse.





The conic sections can also be characterised as follows by comparison of areas.

For this purpose, the equations are considered in the form $y^2 = 2px$ (parabola), $y^2 = 2px \pm \frac{p}{a}x^2$ (hyperbola or ellipse). If you choose a point $P(x,y)$ on the upper branch of the parabola as the corner point of a square with the side length *y*, then this square (blue) was equal in area to a rectangle (orange) with the side lengths *x* and 2*p*. This side length 2*p* is just the length of the chord through the focal point of the conic section. This was also known as the *latus rectum* (transverse line), and the axis *a* is known as the *latus transversum*.

If the point $P(x,y)$ lies on the hyperbola or ellipse, then the rectangle with area 2*px* must be enlarged or reduced by the amount $\frac{p}{a}x^2$ so that it was equal in area to the square with side length *y*.

The terms *parabola, hyperbola* and *ellipse* are derived from APOLLONIUS: the Greek word *paraballein* means "to compare", *hyperballein* "to exceed" and *elleipein* "to lack".

In the first four books of the *Conics* APOLLONIUS could still draw on the knowledge of his predecessors, but in books V to VIII he developed the theory further: shortest and longest distances from a point outside, investigation of the normal and sub-normal, determination of the centre of curvature and circle of curvature, conjugate diameters, special construction tasks.

It was known from various sources that APOLLONIUS has also intensively dealt with the question of which constructions were possible exclusively with ruler and compass.

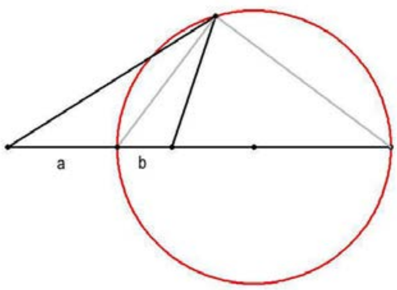
He also wrote a number of other works which have been lost both in the original and in the Arabic translation and they were partly reconstructed in the 17th century.

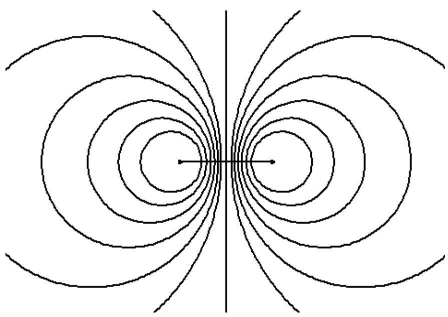
For example:

- *De Locis Planis*: This collection of results about geometric loci, which are exclusively straight lines and circles, fascinated the young PIERRE DE FERMAT so much that he tried to reconstruct them in 1629: *Apollonii Pergaei libri duo de locis planis restituti*.

Usually a circle was defined as a set of points that have the same distance from a certain point *M*. APOLLONIUS proved that it can also be defined as a set of points whose distances from two given points have a constant ratio *a* : *b*. To prove this, he used the theorem about the ratio of one side to the bisector of the opposite point and the theorem of THALES.

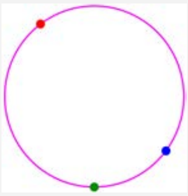
The figure on the right side shows APOLLONIAN circles with different constant ratios – if the ratio is reversed, the circle reflected at the mid-perpendicular of the line is obtained.



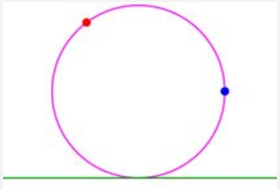


- *De Tactionibus*: You are given three geometrical objects (points, straight lines or circles) and what is sought is a circle that passes through the given points or touches the given straight lines or circles. FRANÇOIS VIÈTE (FRANCISCUS VIETA) published a reconstruction of the work in 1600 under the title *Apollonius Gallus* (he thus calls himself the APOLLONIUS of France).

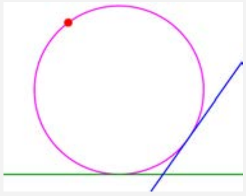
In total there are ten possible cases with a unique solution.



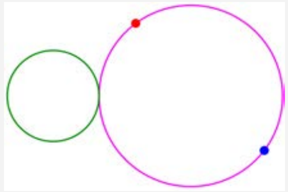
given: three points



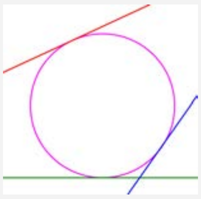
given: two points and one line



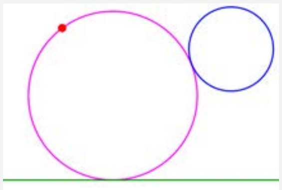
given: two lines and one point



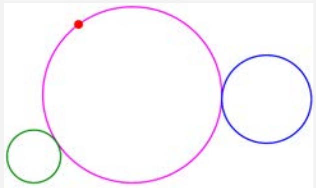
given: two points and one circle



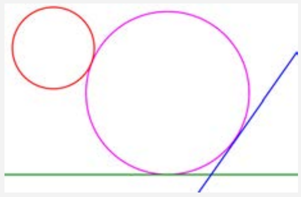
given: three lines



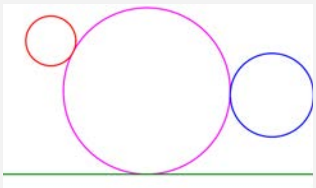
given: one point, one line, one circle



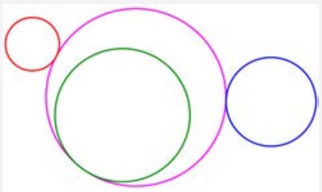
given: two circles and one point



given: two lines and one circle



given: two circles and one line

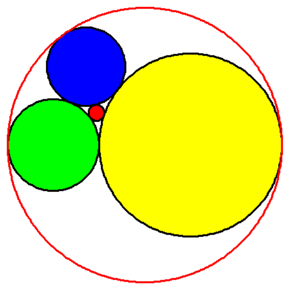


given: three circles

DESCARTES described the solution of the last case (APOLLONIUS's *tangent problem*) by means of an equation:

$$2 \cdot (k_1^2 + k_2^2 + k_3^2 + k_4^2) = (k_1 + k_2 + k_3 + k_4)^2,$$

where $k_i = \pm \frac{1}{r_i}$ (the *curvatures* of the circles).



APOLLONIUS also made a remarkable contribution to astronomy. In accordance with ARISTOTLE's dogma that celestial bodies must move on circular orbits, he succeeded in reconciling this with the observation of the planets' regressions by specifying small circles (epicycles) for the planetary orbits, the centres of which in turn move on circular orbits.

This was the basis for the geocentric world view of HIPPARCHUS (190 – 120 BC) and CLAUDIUS PTOLEMEUS (100 – 180 AD). NIKOLAUS COPERNICUS also still needed epicycles to reconcile his observations with his heliocentric world view. Finally the realisation by JOHANNES KEPLER that the planets move on ellipses and not on circles made these complicated approaches superfluous.

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https://www.spektrum.de/wissen/apollonius-von-perge-262-190-v-chr/1146206
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