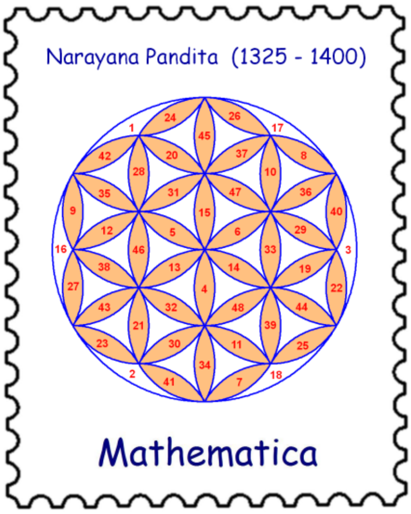
NARAYANA PANDITA (about 1325 – 1400)

by HEINZ KLAUS STRICK, Germany



In 1688, the French ambassador to the Kingdom of Siam, SIMON DE LA LOUBÈRE, published a book about his experiences after his return from East Asia; it was also published in English in 1693.

In one of the chapters, he introduced what he called the *Siamese method* of creating odd-order magic squares.

In fact, this method had been described more than 300 years earlier by NARAYANA PANDITA (*Pandita* in Sanskrit means "scholar").

Almost nothing is known about the life of this mathematician – except that he published two books: *Bijaganita-Vatamsa*, a book on algebra, and in 1356 his major work *Ganita-kaumudi* (literally: Moonlight of Mathematics), which comprised 14 chapters.

The last of these chapters, entitled *Bhadraganita*, dealt with magic squares and figures. The purpose of studying magic figures, according to NARAYANA, was to construct a *yantra* (a geometric diagram to be used for meditation) *to destroy the ego of bad mathematicians and promote the pleasure of good mathematicians.*

The "Siamese" method can be described as follows:

You start by writing the number 1 in the middle box of the top row, then from there diagonally to the top right continuously the next natural numbers.

(1) *When you reach the top edge, you write the next number in a field of the bottom row in the next column.*
(2) *When you reach the right-hand edge, enter the next number in a field of the leftmost column in the next row.*
(3) *If you reach a field that is already occupied or is the upper right corner of the square, you continue the procedure in the field below.*

The following figure (from *Mathematik ist wunderwunderschön*, Springer) shows the method for a 5×5-square (with additional auxiliary fields).

			2		
		1			
	5				
4					4
				3	
			2		

				9	
		1	8		
	5	7			
4	6				
				3	10
			2	9	

					16
		1	8	15	
	5	7	14	16	
4	6	13			
10	12			3	
11			2	9	

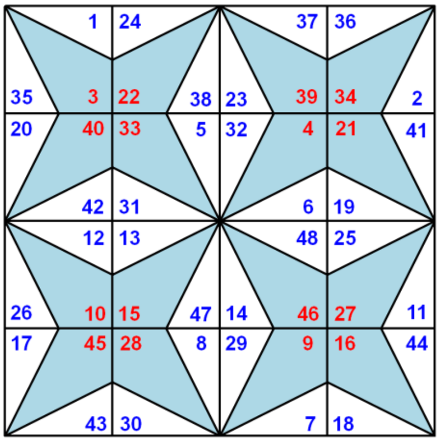
	18				
17		1	8	15	17
	5	7	14	16	
4	6	13	20		
10	12	19		3	
11	18		2	9	

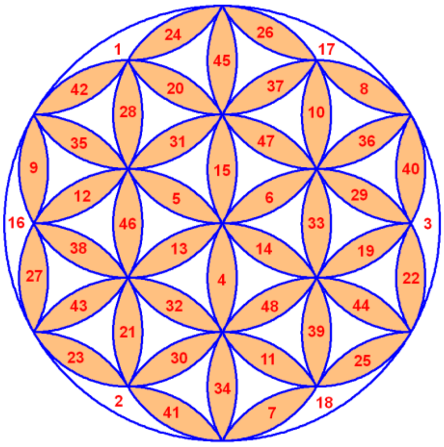
17		1	8	15	
	5	7	14	16	23
4	6	13	20	22	
10	12	19	21	3	
11	18		2	9	

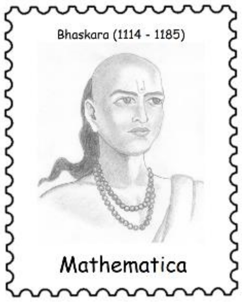
		25			
17	24	1	8	15	
23	5	7	14	16	
4	6	13	20	22	
10	12	19	21	3	
11	18	25	2	9	

NARAYANA's explanations include various methods for constructing magic squares of any order, including all the possibilities for magic squares of order 4.

Finally, he presents special geometric shapes with magical properties, including the *diamond lotus*, where four numbers in a row each add up to 98, eight numbers in a row add up to 196 and each partial square with twelve fields adds up to 294 (cf. fig. left), as well as the *inscribed lotus*, where each flower also has the magic sum 294 (cf. fig. right).







In his work *Ganita-kaumudi*, NARAYANA describes himself as *a humble navigator in the ocean of mathematics*. The book gives an impressive overview of the state of knowledge of the mathematicians of the Indian subcontinent at the end of the Middle Ages.

Some of this had previously been published by BHASKARACHARYA (BHASKARA II, 1114-1185), on whose main work *Līlāvatī* (The Beautiful) NARAYANA had written an extensive commentary. However, *Ganita-kaumudi* also contains a wealth of new material.

In the first chapter, NARAYANA gives an overview of the common measures of weight, length, area and space, of types of calculation (up to and including the method of taking a cube root) and of types of equations. In order to check the result of a multiplication, he recommends comparing the remainders of the factors (with regard to division by any number) with the corresponding remainders of the result – just as we know from "casting out nines".

When dealing with roots, amazing skills are revealed, e.g.

$$\frac{\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60}}{\sqrt{5}+\sqrt{3}+\sqrt{2}}=\ldots=\frac{\sqrt{2100}+\sqrt{1800}+\sqrt{1260}+\sqrt{1080}}{\sqrt{60}+\sqrt{36}}=\ldots=\sqrt{35}+\sqrt{30}$$

The second chapter deals with tasks that lead to linear equations, such as mixing and interest tasks as well as motion tasks.

Example: *Two travellers go from two places A_1 and A_2, which have a distance d from each other, towards each other at the same time with the velocities v_1 and v_2. At what time and at what place do they meet? From the destination, they immediately return to their starting points. When and where do they meet a second time?*

The third chapter deals with sequences and series. Apart from the typical tasks for arithmetic and geometric sequences, it also contains sum formulae for natural numbers, for their squares and for their third powers.

The sum sequence of triangular numbers is also examined:

$$\sum\sum r = \sum \frac{r\cdot(r+1)}{2} = \frac{n\cdot(n+1)\cdot(n+2)}{1\cdot 2\cdot 3}$$

and this can be generalised to the result: $\sum ... \sum\sum r = ... = \frac{n\cdot(n+1)\cdot(n+2)\cdot ...\cdot(n+k)}{1\cdot 2\cdot 3\cdot ...\cdot(k+1)}$.

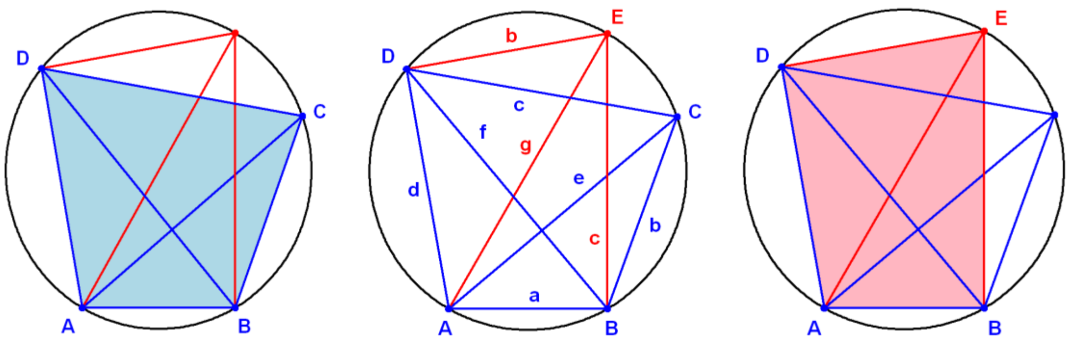
Chapter 4 is the most comprehensive chapter of the book; it comprises 149 rules and 94 examples of geometric problems, including a number of approximation formulae for circular figures.

Remarkable is a formula newly developed by NARAYANA for determining the area of a cyclic quadrilateral with the help of a so-called third diagonal:

From the cyclic quadrilateral *ABCD*, the cyclic quadrilateral *ABED* with the diagonals *f* and *g* is obtained by exchanging the sides *b* and *c*. The area of the two quadrilaterals is the same.

The areas of the two quadrilaterals match, since according to BRAHMAGUPTA's formula the area depends only on the length of the four sides of the cyclic quadrilateral:

$A = \sqrt{(s-a)\cdot(s-b)\cdot(s-c)\cdot(s-d)}$ with $s = \frac{1}{2}\cdot(a+b+c+d)$.



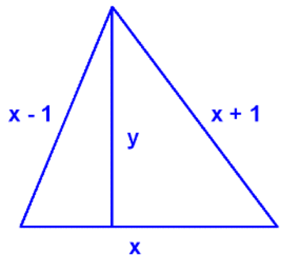
The area of the quadrilateral *ABED* can be calculated as the sum of the areas of the triangle *ABE* with sides *a, c, g* and of the triangle *AED* with sides *b, d, g*:

$$A_{ABED} = A_{ABE} + A_{AED} = \frac{a\cdot c\cdot g}{4R} + \frac{b\cdot d\cdot g}{4R} = \frac{g}{4R}\cdot(a\cdot c + b\cdot d)$$

On the other hand, according to PTOLEMY's theorem, the product of the lengths of the diagonals of a cyclic quadrilateral is equal to the sum of the products of the opposite sides of the cyclic quadrilateral, i.e. $a\cdot c + b\cdot d = e\cdot f$.

Therefore it follows that: $A_{ABCD} = A_{ABED} = \frac{g}{4R}\cdot(a\cdot c + b\cdot d)$, so $A_{ABCD} = \frac{e\cdot f\cdot g}{4R}$.

One of the tasks deals with special triangles whose side lengths are natural numbers and which differ only by one unit; the length of the height on the base side is also supposed to be a natural number.

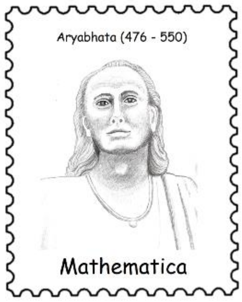


NARAYANA recognises that the left section of the base side must have the length $\frac{1}{2}\cdot x - 2$, the right section the corresponding length $\frac{1}{2}\cdot x + 2$, because according to the theorem of PYTHAGORAS, the following applies to the two partial triangles:

$(x-1)^2 - (\frac{1}{2}\cdot x - 2)^2 = y^2 = (x+1)^2 - (\frac{1}{2}\cdot x + 2)^2$, i.e., the following applies: $y^2 = \frac{3}{4}\cdot x^2 - 3$.

This equation has an infinite number of solutions: (4;3), (14;12), (52;45), (194;168), (724;627), ...

The next chapters deals with application tasks (digging pits, heaping up grain, calculating heights and distances with the help of shadow lengths, etc.).



(drawings © Andreas Strick)

In chapter 9, the *Kuttaka* method developed by ARYABATA (476-550) for solving Diophantine equations is described in detail and explained with examples.

In Chapter 10, NARAYANA also deals with the solution of what later became known as *PELL's equations* (according to BHASKARACHARYA's method). He explicitly addresses the fact that the pairs of solutions (*a, b*) of equations of the type $Nx^2 + 1 = y^2$ can be used to determine approximate values for the root of a natural number: $\sqrt{N} \approx \frac{b}{a}$.

Example: For the equation $10x^2 + 1 = y^2$ we find the pairs of solutions (6, 19), (228, 721), (8658, 27379), etc.

Therefore: $\sqrt{10} \approx \frac{19}{6} = 3.1\overline{6}$; $\sqrt{10} \approx \frac{721}{228} = 3.162280...$; $\sqrt{10} \approx \frac{27379}{8658} = 3.162277...$

In chapter 11, NARAYANA deals with the decomposability of a natural number *n* (which is not a square number) into factors. In doing so, he develops a method based on the same idea that PIERRE DE FERMAT described in 1643 in a letter to MERSENNE – 300 years later.

The aim of the investigation is to represent the number *n* under consideration as the difference between two square numbers x^2 and y^2: From $n = y^2 - x^2$ follows $n = (y-x) \cdot (y+x)$; the natural numbers $y-x$ and $y+x$ can then in turn be examined for decomposability according to the same procedure.

In order to check whether a natural number *n* can be represented as a product of two natural numbers, one makes the approach $n = a^2 + r$, where a^2 is the next-smallest square number (and correspondingly $(a+1)^2$ the next-largest square number).

If $(2a+1) - r$ is a square number b^2, then we have $n + b^2 = (a^2 + r) + (2a + 1 - r) = (a+1)^2$ and thus $n = (a+1)^2 - b^2 = (a+b+1) \cdot (a-b+1)$.

If there is no square number $2a+1-r$, then one goes to the next larger square number. This is done by adding $2a+3$ (= difference $(a+2)^2 - (a+1)^2$ to the next square number) and so on.

Example: $n = 527 = 22^2 + 43$, so $(2a+1) - r = (2 \cdot 22 + 1) - 43 = 45 - 43 = 2$; this is not a square number. Further: $(2a+3) + (2a+1) - r = (2 \cdot 22 + 3) + (2 \cdot 22 + 1) - 43 = 47 + 45 - 43 = 49 = 7^2$, i.e. $(2a+3) + (2a+1) - r = (4a+4) - r = b^2$ and thus $n + b^2 = (a^2 + r) + (4a+4) - r = (a+2)^2$, thus $n = (a+2)^2 - b^2 = (a+b+2) \cdot (a-b+2) = (22+7+2) \cdot (22-7+2) = 31 \cdot 17$.

In chapter 12, NARAYANA takes up a topic that MAHAVIRA (ca. 800-870) had already dealt with: What are the possibilities of representing the number 1 as a sum of reciprocals of integers?

In addition to the – later so-called – FIBONACCI algorithm, MAHAVIRA had also discovered representations with the help of special partial sequences:

$$\frac{1}{2}+\left(\frac{1}{3^1}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}\right)+\frac{1}{2\cdot 3^n}=1 \text{ and } \left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(2n-1)\cdot 2n}\right)+\frac{1}{2n}=1.$$

NARAYANA recognises, among other things, that the following relationship is true in general:

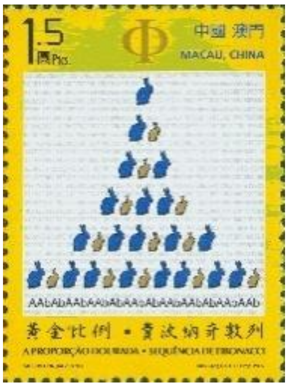
$$\frac{(k_2-k_1)\cdot k_1}{k_2\cdot k_1}+\frac{(k_3-k_2)\cdot k_1}{k_3\cdot k_2}+\ldots+\frac{(k_n-k_{n-1})\cdot k_1}{k_n\cdot k_{n-1}}+\frac{1\cdot k_1}{k_n}=1.$$

The penultimate chapter, which is also very extensive, contains 97 rules and 45 examples on various combinatorial problems, including the number of possible permutations. In this context, NARAYANA develops an algorithm with which one can systematically generate all permutations of objects.

This chapter contains, among other things, the so-called *cow problem* (OEIS A000930), which has a similar structure to FIBONACCI 's *rabbit problem*:

One cow gives birth to a calf every year. Then, starting with the fourth year, each calf also gives birth to a calf at the beginning of each year. How many cows and calves are there in total after 20 years?

The problem can be solved by applying the recursion formula $a(n)=a(n-1)+a(n-3)$ with the initial values $a(0)=a(-1)=a(-2)=1$; it follows that: $a(20)=2745$. NARAYANA calculates the number using binomial coefficients.



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